3 — Limit and Continuity

3.1 Definition of Limit

Definition 3.1.1 Formal Definition of Limit Let f be a function defined on an open interval containing a, with the possible exception of a itself. Then the limit of $f(x)$ as x approaches a is the number L, written as

$$\lim_{x\to a} f(x) = L \quad \text{iff}$$

for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if

$$0 < |x-a| < \delta, \quad then \quad |f(x)-L| < \varepsilon.$$

A Geometric Interpretation

Let $\varepsilon > 0$ be given. Draw the lines $y = L+\varepsilon$ and $y = L-\varepsilon$. Since $|f(x)-L| < \varepsilon$ is equivalent to $L-\varepsilon < x < L+\varepsilon$, $\lim_{x\to a} f(x) = L$ exists provided that we can find a number δ such that if we restrict x to lie in the interval $(a-\delta, a+\delta)$ with $x \neq a$, then the graph of $y = f(x)$ lies inside the band of width 2ε determined by the lines $y = L-\varepsilon$ and $y = L-\varepsilon$. (See Figure 3.1) You can see from Figure 3.1 that once a number $\delta > 0$ has been found, then any number smaller than d will also satisfy the requirement.

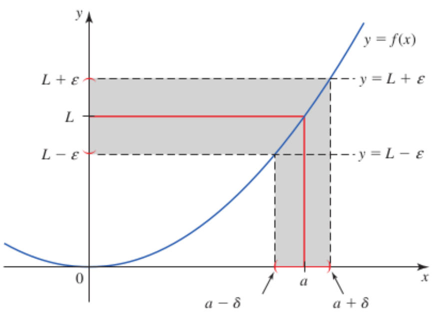


Figure 3.1:

1. δ is a function of ε
2. δ is not unique
3. For all $\delta' < \delta$ it is true

Example 3.1 Show that

$$\lim_{x\to 4}(3x-4) = 8$$

Proof: Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$0 < |x-4| < \delta \quad \Rightarrow \quad |f(x)-8| < \varepsilon$$

Consider

$$\begin{aligned} |f(x)-8| < \varepsilon &\Rightarrow |(3x-4)-8| < \varepsilon \\ &\Rightarrow |3x-12| < \varepsilon \\ &\Rightarrow |x-4| < \frac{\varepsilon}{3} \end{aligned}$$

Now choose $\delta = \frac{\varepsilon}{3}$. Thus

$$0 < |x-4| < \delta \quad \Rightarrow \quad |(3x-4)-8| = |3x-12| = 3|x-4| < 3\frac{\varepsilon}{3} = \varepsilon$$

Example 3.2 Show that $\lim_{x\to 2}(x^2-1) = 3$

Proof: Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$0 < |x-2| < \delta \quad \Rightarrow \quad |f(x)-3| < \varepsilon$$

Consider

$$\begin{aligned} |f(x)-3| < \varepsilon &\Rightarrow |(x^2-1)-3| < \varepsilon \\ &\Rightarrow |x^2-4| < \varepsilon \\ &\Rightarrow |(x-2)(x+2)| < \varepsilon \end{aligned}$$

Let $\delta = 1$, thus,

$$\begin{aligned} |x-2| < 1 \quad &\Rightarrow \quad -1 < x-2 < 1 \\ &\Rightarrow \quad 3 < x+2 < 5 \\ Hence, \quad & |x+2| < 5 \end{aligned}$$

Thus we have

$$|x^2-4| = |x+2||x-2| < 5|x-2| < \varepsilon \Rightarrow |x-2| < \frac{\varepsilon}{5}$$

Now choose $\delta = \min(\frac{\varepsilon}{5}, 1)$
Thus

$$0 < |x-2| < \delta \quad \Rightarrow \quad |(x^2-1)-3| < 5|x-2| < 5\frac{\varepsilon}{5} = \varepsilon$$

Example 3.3 Using $\varepsilon - \delta$ definition of limit prove that

1. $\lim_{x\to 3}(x^2+x) = 12$
2. $\lim_{x\to -3}(x^2+x) = 6$
3. $\lim_{x\to 3}\frac{2}{x+3} = \frac{1}{3}$
4. $\lim_{x\to 9}(2+\sqrt{x}) = 5$

Proof. 1. Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$0 < |x-3| < \delta \quad \Rightarrow \quad |(x^2+x)-12| < \varepsilon$$

Consider

$$\begin{aligned} |f(x)-12| < \varepsilon \quad &\Rightarrow \quad |(x^2+x)-12)| < \varepsilon \\ &\Rightarrow \quad |(x+4)(x-3)| < \varepsilon \\ &\Rightarrow \quad |x+4||x-3| < \varepsilon \end{aligned}$$

Take $\delta = 1$, then

$$\begin{aligned} |x-3| < \delta \quad &\Rightarrow \quad |x-3| < 1 \\ &\Rightarrow \quad -1 < x-3 < 1 \\ &\Rightarrow \quad 2 < x < 4 \\ &\Rightarrow \quad 6 < |x+4| < 8 \end{aligned}$$

Thus we have

$$|f(x)-12| = |x+4||x-3| < 8|x-3| < \varepsilon \quad \text{iff} \quad |x-3| < \frac{\varepsilon}{8}$$

Now choose $\delta = min(1, \frac{\varepsilon}{8})$

Thus,

$$0 < |x-3| < \delta \Rightarrow |x^2+x-12| = |x+4||x-3| < 8\frac{\varepsilon}{8} = \varepsilon$$

2. Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$0 < |x-(-3)| < \delta \quad \Rightarrow \quad |(x^2+x)-6| < \varepsilon$$

Consider

$$\begin{aligned} |f(x)-6| < \varepsilon \quad &\Rightarrow \quad |(x^2+x)-6| < \varepsilon \\ &\Rightarrow \quad |(x-2)(x+3)| < \varepsilon \\ &\Rightarrow \quad |x+3||x-2| < \varepsilon \end{aligned}$$

Take $\delta = 1$, then

$$\begin{aligned} |x+3| < \delta \quad &\Rightarrow \quad |x+3| < 1 \\ &\Rightarrow \quad -1 < x+3 < 1 \\ &\Rightarrow \quad -4 < x < -2 \\ &\Rightarrow \quad -6 < x-2 < -4 \\ &\Rightarrow \quad 4 < 2-x < 6 \\ &\Rightarrow \quad |2-x| < 6 \end{aligned}$$

Thus we have

$$|f(x)-6| = |x+3||x-2| < 6|x+3| < \varepsilon \quad \text{iff} \quad |x+3| < \frac{\varepsilon}{6}$$

Now choose $\delta = min(1, \frac{\varepsilon}{6})$

Thus,

$$0 < |x-3| < \delta \Rightarrow |x^2+x-6| = |x+3||x-2| < 6\frac{\varepsilon}{6} = \varepsilon$$

$$\therefore \lim_{x\to -3}(x^2+x) = 6$$

3. Let $\varepsilon > 0$ be given. We must find δ such that

$$0 < |x-3| < \delta \Rightarrow \left|f(x) - \frac{1}{3}\right| < \varepsilon$$

Consider

$$\begin{aligned} \left|f(x) - \frac{1}{3}\right| < \varepsilon &\Rightarrow \left|\frac{2}{x+3} - \frac{1}{3}\right| < \varepsilon \\ &\Rightarrow \left|\frac{6-(x+3)}{3(x+3)}\right| < \varepsilon \\ &\Rightarrow \frac{1}{3}\left|\frac{x-3}{x+3}\right| < \varepsilon \end{aligned}$$

Take $\delta = 1$, then

$$\begin{aligned} |x-3| < \delta &\Rightarrow |x-3| < 1 \\ &\Rightarrow -1 < x-3 < 1 \\ &\Rightarrow 2 < x < 4 \\ &\Rightarrow 5 < |x+3| < 7 \\ &\Rightarrow \frac{1}{7} < \frac{1}{|x+3|} < \frac{1}{5} \end{aligned}$$

Thus we have

$$\frac{1}{3}|x-3|\frac{1}{|x+3|} < \frac{1}{15}|x-3| < \varepsilon \quad \text{iff} \quad |x-3| < 15\varepsilon$$

Now choose $\delta = min(1, 15\varepsilon)$

Thus,

$$0 < |x-3| < \delta \quad \Rightarrow \quad |\frac{2}{x+3} - \frac{1}{3}| = \frac{1}{3}|\frac{x-3}{x+3}| < \frac{1}{15}15\varepsilon = \varepsilon$$

4. Let $\varepsilon > 0$ be given. We must find δ such that

$$0 < |x-9| < \delta \Rightarrow |f(x) - 5| < \varepsilon$$

Consider

$$\begin{aligned} |f(x) - 5| < \varepsilon &\Rightarrow |(2+\sqrt{x}) - 5| < \varepsilon \\ &\Rightarrow |\sqrt{x} - 3| < \varepsilon \\ &\Rightarrow |\frac{x-9}{\sqrt{x}+3}| < \varepsilon \end{aligned}$$

Take $\delta = 1$, then

$$\begin{aligned} |x-9| < \delta &\Rightarrow |x-9| < 1 \\ &\Rightarrow -1 < x-9 < 1 \\ &\Rightarrow 8 < x < 10 \\ &\Rightarrow \sqrt{8}+3 < \sqrt{x}+3 < \sqrt{10}+3 \\ &\Rightarrow \frac{1}{\sqrt{10}+3} < \frac{1}{|\sqrt{x}+3|} < \frac{1}{\sqrt{8}+3} \end{aligned}$$

Thus we have

$$|x-9|\frac{1}{|\sqrt{x}+3|} < \frac{1}{\sqrt{8}+3}|x-9| < \varepsilon \quad iff \quad |x-9| < (\sqrt{8}+3)\varepsilon$$

Now choose $\delta = min(1, (\sqrt{8}+3)\varepsilon)$

Thus,

$$\begin{aligned} 0 < |x-9| < \delta \Rightarrow |(2+\sqrt{x}-5| &= \left|\frac{x-9}{\sqrt{x}+3}\right| \\ &< \frac{1}{\sqrt{8}+3}(\sqrt{8}+3)\varepsilon = \varepsilon \end{aligned}$$

■

3.2 Basic Limit Theorems

Theorem 3.2.1 If m and b are any constants, then $\lim_{x \to a}(mx+b) = ma+b$

Theorem 3.2.2 If c is a constant, then for any number a, $\lim_{x \to a} c = c$

Theorem 3.2.3 If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then;

1. $\lim_{x \to a}(f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$
2. $\lim_{x \to a}(f(x).g(x)) = \lim_{x \to a} f(x). \lim_{x \to a} g(x) = L.M$
3. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{M}$ provided that $M \neq 0$

Theorem 3.2.4 If $\lim_{x \to a} f(x) = L$ and n is any positive integer, then

$$\lim_{x \to a}[f(x)]^n = L^n$$

Theorem 3.2.5 If $\lim_{x \to a} f(x) = L$, then

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$$

if $L > 0$ and n is any positive integer, or if $L \leq 0$ and n is a positive odd integer

Theorem 3.2.6 If f and g are functions such that $\lim_{x \to a} g(x) = L$ and $\lim_{x \to a} f(x) = f(L)$, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L)$$

Example 3.4 Evaluate the following limit

(a) $\lim_{x \to -1}(3x^2 - 2x + 1)^3$

(b) $\lim_{x \to 1} \dfrac{3x^3 - 5x^2 - 1}{2x - 3}$

(c) $\lim_{x \to 2} \dfrac{2x^2 - 5x + 2}{5x^2 - 7x - 6}$

(d) $\lim_{x \to 9} \dfrac{x - 9}{\sqrt{x} - 3}$

(e) $\lim_{x \to 0} \dfrac{\sqrt{x+1} - \sqrt{1-x}}{x}$

(f) $\lim_{x \to 8} \dfrac{x^{2/3} + 3\sqrt{x}}{4 - 16/x}$

(g) $\lim_{x \to 5} \sqrt[3]{3x^2 - 4x + 9}$

(h) $\lim_{x \to 9} \dfrac{x^3 - 1}{x - 1}$

(i) $\lim_{x \to 1} \dfrac{3x - 3}{\sqrt[3]{x+7} - 2}$

Solution: (i) Let $t = \lim_{x \to 1} \sqrt[3]{x+7} \implies t^3 = x + 7 \implies x = t^3 - 7$
Observe that as x approaches 1, t approaches 2. Therefore

$$\begin{aligned}
\lim_{x \to 1} \frac{3x - 3}{\sqrt[3]{x+7} - 2} &= \lim_{t \to 2} \frac{3(t^3 - 7) - 3}{t - 2} = \lim_{t \to 2} \frac{3t^3 - 24}{t - 2} \\
&= \lim_{t \to 2} \frac{3(t-2)(t^2 + 2t + 4)}{t - 2} = \lim_{t \to 2} 3(t^2 + 2t + 4) = 36
\end{aligned}$$

3.3 One sided Limits

Definition 3.3.1
1. Let f be a function which is defined at every number in some open interval (a,c). Then the limit of $f(x)$, as x approaches a from the right, is L, written
$$\lim_{x\to a^+} f(x) = L$$
if for any $\varepsilon > 0$, however small, there exists a $\delta > 0$ such that,
$$|f(x) - L| < \varepsilon \text{ whenever } 0 < x - a < \delta$$

2. Let f be a function which is defined at every number in some open interval (d,a). Then the limit of $f(x)$, as x approaches a from the left, is L, written
$$\lim_{x\to a^-} f(x) = L$$
if for any $\varepsilon > 0$, however small, there exists a $\delta > 0$ such that,
$$|f(x) - L| < \varepsilon \text{ whenever } -\delta < x - a < 0$$

Theorem 3.3.1 If f is defined throughout an open interval containing a, except possibly at a itself, then $\lim_{x\to a} f(x) = L$ if and only if both $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

■ **Example 3.5** Find $\lim_{x\to 2^+} (1 + \sqrt{x-2})$ ■

■ **Example 3.6** Suppose $f(x) = \dfrac{|x|}{x}$ if $x \neq 0$ and $f(0) = 1$. Find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$. What is $\lim_{x\to 0} f(x)$? ■

■ **Example 3.7** Suppose $f(x) = \begin{cases} x+3 & \text{if} \quad x \neq 1 \\ 2, & \text{if} \quad x = 1 \end{cases}$ ■

■ **Example 3.8** Suppose $f(x) = \begin{cases} -x+3 & \text{if} \quad x \leq 2 \\ \sqrt{x-2}, & \text{if} \quad x \geq 2 \end{cases}$ ■

The Squeeze Theorem

Theorem 3.3.2 The Squeeze Theorem
Suppose that $f(x) \leq g(x) \leq h(x)$ for all x in an open interval containing a, except possibly at a, and
$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$
Then
$$\lim_{x\to a} g(x) = L$$

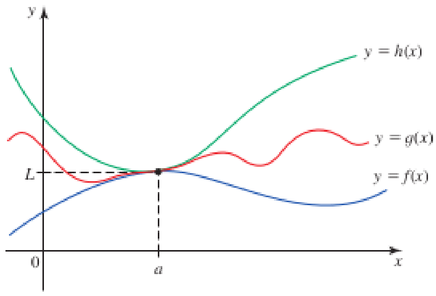


Figure1: An illustration of the Squeeze Theorem

Example 3.9 Find $\lim_{x\to 0} x^2 \sin\frac{1}{x}$

Solution: Since $-1 \le \sin t \le 1$ for every real number t, we have

$$-1 \le \sin\frac{1}{x} \le 1 \quad for\,every \quad x \ne 0$$

Therefore

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2, \quad x \ne 0$$

Let $f(x) = -x^2$, $g(x) = x^2\sin\frac{1}{x}$, $and\, h(x) = x^2$.Then

$$f(x) \le g(x) \le h(x).$$

Since

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}(-x^2) = 0 \quad and \quad \lim_{x\to 0} h(x) = \lim_{x\to 0} x^2 = 0$$

The squeeze theorem implies that

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

Theorem 3.3.3

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Proof: First, suppose that $0 < x < \frac{\pi}{2}$. The Figure below shows a sector of a circle of radius 1.
The area A of a sector with radius r and a central angle of θ radians can be derived by:

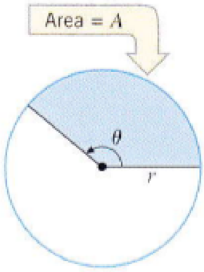
$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}\left[\frac{\text{area of the sector}}{\text{area of the circle}} = \frac{\text{central angle of the sector}}{\text{centar angle of the circle}}\right]$$

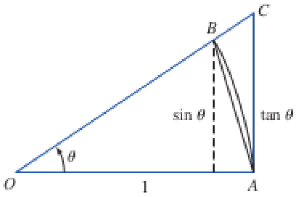
From the figure we see that

$$\text{Area of } \triangle OAB = \frac{1}{2}\text{base.height} = \frac{1}{2}(1)\sin\theta = \frac{1}{2}\sin\theta$$

$$\text{Area of sector } OAB = \frac{1}{2}r^2\theta = \frac{1}{2}\theta$$

$$\text{Area of } \triangle OAC = \frac{1}{2}\text{base.height} = \frac{1}{2}(1)\tan\theta = \frac{1}{2}\tan\theta$$





Since $0 <$ Area of $\triangle OAB <$ Area of sector $OAB <$ Area of $\triangle OAC$, we have,

$$0 < \frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

Multiplying through by $(2/\sin\theta)$ and keeping in mind that $\sin\theta > 0$ *and* $\cos\theta > 0$ for $0 < \theta < \pi/2$ we obtain

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

or, upon taking reciprocals,

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Let $f(\theta) = \cos\theta$, $g(\theta) = \dfrac{\sin\theta}{\theta}$, $h(\theta) = 1$ Observe that,

$$\lim_{\theta\to 0} f(0) = \lim_{\theta\to 0}\cos\theta = 1 = \lim_{\theta\to 0} h(\theta)$$

Then the squeeze Theorem implies that

$$\lim_{\theta\to 0} g(\theta) = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Theorem 3.3.4

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$$

Example 3.10 Find

(a) $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{3x}$

(b) $\displaystyle\lim_{x\to 0}\frac{\tan x}{x}$

(c) $\displaystyle\lim_{x\to 0}\frac{2x+1-\cos x}{4x}$

Solution:

(a) $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{3x} = \lim_{x\to 0}\frac{2}{3}\frac{\sin 2x}{2x} = \frac{2}{3}(1) = \frac{2}{3}$

(b) $\displaystyle\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{1}{\cos x}\frac{\sin x}{x} = \frac{1}{\cos 0}(1) = 1$

(c) $\displaystyle\lim_{x\to 0}\frac{2x+1-\cos x}{4x} = \lim_{x\to 0}\frac{2x}{4x} - \frac{\cos x - 1}{4x} = \frac{2}{4} - (1)(0) = \frac{1}{2}$

3.4 Infinite Limits, Limit at infinity and Asymptotes

3.4.1 Infinite limits and Vertical Asymptotes

Definition 3.4.1 Let f be defined on an interval that contains a except possibly at a itself. Then

1. $\lim_{x \to a} f(x) = \infty$ iff for every $M > 0$, there is a $\delta > 0$, such that
$$\text{if } 0 < |x-a| < \delta, \text{ then } f(x) > M$$
2. $\lim_{x \to a} f(x) = -\infty$ iff for every $N > 0$, there is a $\delta > 0$,
$$\text{if } 0 < |x-a| < \delta, \text{ then } f(x) < N$$

Definition 3.4.2 The line $x = a$ is called a vertical asymptote of the graph of $y = f(x)$ if any one of the following limits holds true:
$$\lim_{x \to a^-} f(x) = \pm\infty, \ \lim_{x \to a^-} f(x) = \pm\infty, \ \lim_{x \to a} f(x) = \pm\infty$$

Example 3.11 Show that $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Solution: Let $M > 0$ be given. We want to show that there exists a $\delta > 0$ such that
$$\frac{1}{x^2} > M \text{ whenever } 0 < |x-0| < \delta.$$
To find δ, consider
$$\frac{1}{x^2} > M$$
$$x^2 < \frac{1}{M} \implies |x| < \frac{1}{\sqrt{M}}$$
Choose $\delta = \frac{1}{\sqrt{M}}$. Thus, $|x| < \delta \implies x^2 < \delta^2$

so $\frac{1}{x^2} > \frac{1}{\delta^2} \geq M$

Example 3.12 (a) $\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$ (c) $\lim_{x \to 3^+} \frac{x^2+x+2}{x^2-2x-3} = \infty$

(b) $\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$

Therefore, the line $x = 3$ is the vertical asymptote of $f(x) = \frac{2x}{x-3}$

3.4.2 Limit at infinity and Horizontal asymptotes

Definition 3.4.3

1. Let f be defined on an interval (a, ∞), then
$\lim_{x \to \infty} f(x) = L$ iff
for every $\varepsilon > 0$ there is a number $M > 0$ such that if $x > M$, then $|f(x) - L| < \varepsilon$.
2. Let f be defined on an interval $(-\infty, a)$, then
$\lim_{x \to -\infty} f(x) = L$ iff
for every $\varepsilon > 0$ there is a number $M < 0$ such that if $x < M$, then $|f(x) - L| < \varepsilon$.

Definition 3.4.4 The line $y = L$ is called horizontal asymptote of the graph of $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \text{ , or } \lim_{x\to-\infty} f(x) = L$$

Example 3.13 Find

(a) $\lim_{x\to\infty} \frac{1}{x} = 0$

(b) $\lim_{x\to-\infty} \frac{1}{x} = 0$

(c) $\lim_{x\to\infty} \frac{2x^2 - x + 1}{3x^2 + 2x - 1}$

(d) $\lim_{x\to-\infty} \frac{x^2+1}{x-2}$

(e) $\lim_{x\to-\infty} \frac{3x}{\sqrt{x^2+1}}$

Therefore, the line $y = 0$ is the horizontal asymptote of $f(x) = \frac{1}{x}$

Example 3.14 Find the vertical and horizontal asymptotes of the graph of

$$f(x) = \frac{\sqrt{9x^2+1}}{3x-5}$$

Solution:

1. To find vertical asymptote

$$\lim_{x\to\frac{5}{3}^-} \frac{\sqrt{9x^2+1}}{3x-5} = -\infty$$

Thus, the line $x = \frac{5}{3}$ is the vertical asymptote of $f(x)$

2. To find the horizontal Asymptote

$$\begin{aligned} \lim_{x\to\infty} \frac{\sqrt{9x^2+1}}{3x-5} &= \lim_{x\to\infty} \frac{\sqrt{x^2(9+\frac{1}{x^2})}}{3x-5} \\ &= \lim_{x\to\infty} \frac{x\sqrt{9+\frac{1}{x^2}}}{3x-5}, \textit{since } x > 0, \sqrt{x^2} = x \\ &= \lim_{x\to\infty} \frac{x\sqrt{9+\frac{1}{x^2}}}{x(3-\frac{5}{x})} = \lim_{x\to\infty} \frac{\sqrt{9+\frac{1}{x^2}}}{(3-\frac{5}{x})} = \frac{\sqrt{9}}{3} = 1 \end{aligned}$$

Again,

$$\begin{aligned} \lim_{x\to-\infty} \frac{\sqrt{9x^2+1}}{3x-5} &= \lim_{x\to-\infty} \frac{\sqrt{x^2(9+\frac{1}{x^2})}}{3x-5} = \lim_{x\to-\infty} \frac{(-x)\sqrt{9+\frac{1}{x^2}}}{3x-5}, \textit{since } x < 0, \sqrt{x^2} = -x \\ &= \lim_{x\to-\infty} \frac{(-x)\sqrt{9+\frac{1}{x^2}}}{x(3-\frac{5}{x})} = \lim_{x\to-\infty} \frac{(-1)\sqrt{9+\frac{1}{x^2}}}{(3-\frac{5}{x})} = \frac{-\sqrt{9}}{3} = -1 \end{aligned}$$

Thus, the lines $y = 1$ and $y = -1$ are the horizontal asymptote of $f(x)$

Exercise 3.1 Find the asymptote of the following functions

1. $f(x) = \frac{2x}{1-x^2}$
2. $g(x) = \frac{2x-5}{x^3-2x^2+x}$

3.5 Continuity of a function

Definition 3.5.1 A function f is said to be continuous at a if $\lim_{x\to a} f(x) = f(a)$

Note: If f is continuous at a, then

1. $f(a)$ is defined
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

If f is not continuous at a, then we say that f is discontinuous at a.

Example 3.15 Find the points of discontinuity of

1. $f(x) = \frac{x^2-x-2}{x-2}$
2. $f(x) = \begin{cases} \frac{1}{x^2}, & if \quad x \neq 0 \\ 1, & if \quad x = 0 \end{cases}$
3. $f(x) = \frac{2x}{1-x^2}$
4. $f(x) = \begin{cases} \frac{x^2-x-2}{x-2}, & if \quad x \neq 2 \\ 1, & if \quad x = 2 \end{cases}$

3.5.1 One sided continuity

Definition 3.5.2

1. f is said to be continuous from the right at a if
$$\lim_{x\to a^+} f(x) = f(a)$$
2. f is said to be continuous from the left at a if
$$\lim_{x\to a^-} f(x) = f(a)$$

Example 3.16 Let $f(x) = \begin{cases} 2x+1 & \text{if } x < 1 \\ x^2-2 & \text{if } x \geq 1 \end{cases}$
Determine whether f is continuous from the right at 1 and/or from the left at 1.

Solution: Because
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x+1) = 3 \neq f(1) = -1.$$
Hence, f is not continuous from the left at 1
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x+1) = -1 = f(1) = -1.$$
Hence, f is continuous from the right at 1.

(R) A function f is continuous at a if and only if f is simultaneously continuous from the right and from the left at a.

3.5.2 Continuity on Intervals

Definition 3.5.3 A function f is said to be **continuous on an open interval** (a,b) if it is continuous at each point of (a,b). A function f is **continuous on a closed interval** $[a,b]$ if it is continuous on (a,b) and is also continuous from the right at a and from the left at b . A function f is **continuous on a half-open interval** $[a,b)$ or $(a,b]$ if f is continuous on (a, b) and f is continuous from the right at a or f is continuous from the left at b , respectively.

■ **Example 3.17** Show that the function f defined by $f(x) = \sqrt{9-x^2}$ is continuous on the closed interval $[-3,3]$. ■

Solution: We first show that f is continuous on $(-3,3)$. Let a be any number in $(-3,3)$. Then, using the laws of limits, we have

$$\lim_{x\to a} f(x) = \lim_{x\to a} \sqrt{9-x^2} = \sqrt{9-a^2} = f(a)$$

and this proves the assertion.
Next, let us show that f is continuous from the right at -3 and from the left at 3.
Again, by invoking the limit properties, we see that

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \sqrt{9-x^2} = \sqrt{9-3^2} = 0 = f(-3)$$

and

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \sqrt{9-x^2} = \sqrt{9-3^2} = 0 = f(3)$$ and this proves the assertion. Therefore, f is continuous on $[-3,3]$.

Theorem 3.5.1 If f and g are continuous at a, and c is a constant then the following are also continuous at a.

(a) $f \pm g$, (b) cf, (c) fg, (d) $\frac{f}{g}$, if $g(a) \neq 0$

Theorem 3.5.2 The following functions:
Polynomial, Rationals, Root functions, Trigonometric functions, Inverse trigonometric functions, exponential functions and Logarithmic functions are continuous on thier domain.

Theorem 3.5.3 If $\lim_{x\to a} g(x) = b$ and f is continuous at b, then

$$\lim_{x\to a} f(g(x)) = f(b) \text{ i.e., } \lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$$

Theorem 3.5.4 If g is continuous at a and f is continuous at $g(a)$, then fog is continuous at a i.e.,

$$\lim_{x\to a} f(g(x)) = f(g(a))$$

■ **Example 3.18** Find the constant a and b such that

$$f(x) = \begin{cases} ax+3, & if \;\; x>1 \\ 4 & if \;\; x=1 \\ x^2+b & if \;\; x<1 \end{cases}$$

is continuous at $x = 1$ ■

Solution: f is continuous at a iff $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$

$$\begin{aligned}
\lim_{x\to 1^+} f(x) = f(1) &\Rightarrow \lim_{x\to 1^+}(ax+3) = f(1)\\
&\Rightarrow a+3=4 \Rightarrow a=4-3=1\\
&\textit{and}\\
\lim_{x\to 1^-} f(x) = f(1) &\Rightarrow \lim_{x\to 1^-}(x^2+b) = f(1)\\
&\Rightarrow 1+b=4 \Rightarrow b=4-1=3
\end{aligned}$$

Exercise 3.2 Determine whether f is continuous or discontinuous at a.

1. $f(x) = \sqrt{x-2};\quad a=2$
2. $f(x) = \begin{cases} \dfrac{|x-4|}{x-4} & \text{for } x\neq 4 \\ e^{4-x} & \text{for } x=4 \end{cases} \qquad a=4$
3. $f(x) = \begin{cases} x\sin\dfrac{1}{x} & \text{if } x\neq 0 \\ 2 & \text{if } x=0 \end{cases} \qquad a=0$

■

3.5.3 Intermediate Value Theorem (IVT)

Theorem 3.5.5 Intermediate Value Theorem (IVT)
If $f\in C[a,b]$ and k is any number between $f(a)$ and $f(b)$, where $f(a)\neq f(b)$ then there exist a number c in (a,b) for which $f(c)=k$

Theorem 3.5.6 IVT for Locating root
If $f(x)$ is continuous on $[a,b]$, $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one number x_0 in (a,b) such that $f(x_0)=0$.

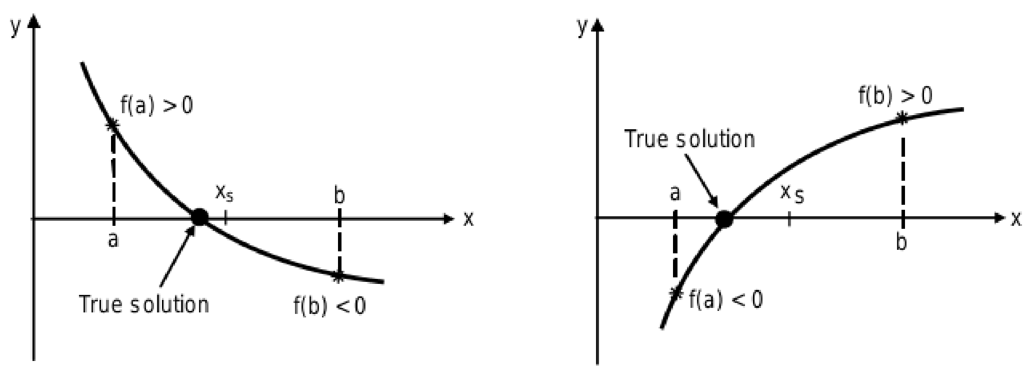


Figure 3.2: Solution of $f(x)=0$ between $x=a$ and $x=b$

■ **Example 3.19** Show that the equation
$$x^5-2x^4-2x^3+8x^2-3x-3$$
has a solution between *1 and 2* ■

Solution: Let

$$f(x) = x^5 - 2x^4 - 2x^3 + 8x^2 - 3x - 3$$

Since f is a polynomial function, then f is continuous on $\mathbb{R}$.

Note that $f(1) = -1$ and $f(2) = 7$

Thus, $f(1) < 0 < f(2)$

i.e., $k = 0$ is between $f(1)$ and $f(2)$

So, by the IVT there is a number c between *1 and 2* such that $f(c) = 0$.

Exercise 3.3 Show that the equation given below has at least one real root.

1. $2x^3 + x^2 - x + 1 = 5$ in $[1,2]$
2. $x + tanx = 1$ in $[0, \frac{\pi}{4}]$
3. $x^5 - x^2 + 2x + 3 = 0$ in $[-1,0]$